A Trilemma for Asset Demand Estimation

William Fuchs Satoshi Fukuda Daniel Neuhann

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Introduction

- Influential literature on the structural estimation of demand functions for financial assets.

- The typical object of interest is the **asset-level price elasticity of demand**

$$\mathcal{E}_{jk} = -\frac{\partial a_j(\vec{a}_{-j}, \vec{p})}{\partial p_k} \times \frac{p_k}{a_j(\vec{a}_{-j}, \vec{p})}.$$

- Most commonly estimated using suitably exogenous supply shocks: mandates, constraints, policies. . . .

- Structural perspective: of interest to the extent that it reveals deep parameters or properties.

Two questions

1. Are asset-level demand elasticities "useful" objects of analysis for asset demand systems?

   Which specific thought experiments and/or structural parameters do they map to?

2. When can asset-level elasticities be identified from observational data?

   Do asset-level supply shocks generate the "right" price variation in equilibrium settings?

A trilemma for observational data

Elasticities are not well-defined and/or cannot be identified if the following conditions jointly hold:

(i) preferences are at least in part defined over cash flows rather than assets.

(ii) prices satisfy no arbitrage.

(iii) identifying variation is based on asset-level supply shocks.

Tension: cross-asset restrictions from (i) and (ii) run counter to basic concept of a demand elasticity.

Exception: the asset menu consists of Arrow securities (for which portfolio restrictions are immaterial).

Framework

- Two dates, $t = 0, 1$. At date 1, one of Z states of the world is realized.
- Investor i has utility function u^i defined over consumption at date 0 and date 1
- Can invest in J assets.
- $y_j(z) \geq 0$ is the payoff of asset j in state z. Y is the $J \times Z$ payoff matrix.
- Investor i's endowment of asset j is e^i_j. Aggregate endowment is E_j.
- p: the vector of asset prices.
- q: vector of state prices (need not be unique).

Minimal properties

Assume: state price q_z is strictly decreasing in the aggregate endowment of state z consumption.

$\Rightarrow$ Let Y_j denote the j-th row of Y. There exists some strictly positive matrix $\mathcal{U}$ such that

$$\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{E}_j} = -\mathcal{U} Y_j.$$

In standard settings, this is guaranteed by strictly concave utility over consumption.

Assume: Prices satisfy no arbitrage, $p = Yq$.

Conceptual considerations

The fundamental challenge

Portfolio choice is built on the idea that investors care about consumption, not assets per sé.

Consequences:

1. Deep preference parameters are only **indirectly** linked to observable asset positions and prices:

$$\boxed{\text{state prices} = \mathcal{H}(\text{asset prices}) \qquad \text{and} \qquad \text{consumption} = \mathcal{G}(\text{asset positions})}$$

2. Portfolio choice problems require restrictions on relative asset prices (i.e., no arbitrage).

Theoretical notions of demand elasticities and identification strategies must reckon with these effects.

A simple representation (in complete markets)

Treat c_0 as the numeraire. Given observed asset prices, each investor solves a three-step problem:

1. Invert payoff matrix Y to determine state prices q:

$$q = Y^{-1} p$$

2. Find the optimal consumption plan $c^i(q)$ as a function of state prices q:

$$\max_{(c_0^i, (c_z^i)_z)} \quad (1-\delta) \cdot u^i(c_0^i) + \delta \sum_{z=1}^{Z} \pi_z \cdot u^i(c_z^i)$$

$$\text{s.t.} \quad c_0^i + \sum_{z=1}^{Z} q_z c_z^i = W^i \quad \text{and} \quad W^i \equiv e_0^i + \sum_{z=1}^{Z} q_z \left(\sum_{j=1}^{J} y_j(z) e_j^i \right).$$

3. Implement the consumption plan by **bundling assets** together, $a^i = \mathcal{G}^{-1}(c^i)$.

What about incomplete markets?

Same basic problem, except q is not unique and we have constraints on feasible consumption plans.

Not central to our arguments, except for extreme forms of incompleteness.

An elasticity is a thought experiment

The demand elasticity is based on a **thought experiment**:

" What would an investor do if a single asset price p_j changes but all other asset prices remain fixed?"

In principle, this experiment can be well-defined in the theory:

1. Figure out the **state price changes induced by the price shock**:

$$\frac{\partial \boldsymbol{q}}{\partial p_j} = \left(\boldsymbol{Y}^{-1}\right)_j$$

2. Figure out the desired change in the optimal consumption plan given new state prices.

3. Compute the percentage change in implied asset-level holdings given new consumption plan.

The role of no arbitrage

1. If the single price change triggers an arbitrage, the decision problem is not well-defined.

2. If we require no arbitrage, it may not be feasible to hold all other prices fixed.

   We must vary state prices, which can in turn affect other asset prices

No arbitrage versus ceteris paribus

Proposition 1. If no arbitrage holds, then a shock to the price of asset j does not imply a change to the price of some other asset iff there exists a z such that $y_j(z) > 0$ and $y_k(z) = 0$ for all $k \neq j$.

$\Rightarrow$ Ceteris paribus condition cannot hold unless assets have **unique exposures to priced risks.**

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Implication. Demand elasticities are consistent with NA only under stringent conditions:

(i) If markets are complete and no redundant assets, Y must be diagonal (up to permutation).

(ii) If markets are incomplete, Y must include a diagonal matrix (up to permutation).

For redundant assets, individual price changes directly trigger arbitrages.

Identification from observational data

Required versus equilibrium price variation

The demand elasticity **presupposes a specific vector of state price changes**, say Δq^{DE} (from before).

Under equilibrium play, supply shocks induce their own vector of state price changes, say Δq^{SS}.

To identify the asset-level elasticity based on supply variation, **these should be proportional.**

1. Under which conditions is Δq^{DE} proportional to Δq^{SS}?
2. When are they of the *same sign*?

Do supply shocks create the right state price variation?

Proposition 2. $\Delta q^{SS} = k \cdot \Delta q^{DE}$ only if the payoff matrix is the identity matrix, $Y = \mathbf{I}$.

"Proof." Effects of supply shock proportional to Y, hypothetical price variation proportional to Y^{-1}.

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Proposition 3. With complete markets, $\mathrm{sign}(-\Delta q^{SS}) = \mathrm{sign}(\Delta q^{DE})$ iff $Y = \mathbf{I}$.

Incomplete markets work similarly: require Y to include a diagonal matrix up to permutation.

Proof. Plemmons and Cline (PAMS, 1972).

Example: A simple equilibrium economy

- Representative investor with log utility. No discounting. Date-0 endowment equal to 1.
- Two equally likely states and two assets: $j, z \in \{g, r\}$. Aggregate supply $E_g = 1 + s_g$ and $E_r = 1$.

	State g	State r
Asset g	$\frac{1}{2}(1+\epsilon)$	$\frac{1}{2}(1-\epsilon)$
Asset r	$\frac{1}{2}(1-\epsilon)$	$\frac{1}{2}(1+\epsilon)$

- Standard optimality condition shows that we **need to get relative state prices right**:

$$q_z = \frac{1}{2}\frac{c_0}{c_z} \qquad \Rightarrow \qquad \frac{c_r}{c_g} = \frac{q_g}{q_r}.$$

Δq^{DE}: State price changes given a hypothetical asset price change

We can back out implied state prices from asset prices:

$$\begin{pmatrix} q_g \\ q_r \end{pmatrix} = \frac{1}{8\epsilon} \begin{pmatrix} (1+\epsilon)p_g - (1-\epsilon)p_r \\ -(1-\epsilon)p_g + (1+\epsilon)p_r \end{pmatrix}.$$

In the **thought experiment** where we vary p_g exogenously, induced state price changes are

$$\frac{\partial}{\partial p_g} \begin{pmatrix} q_g \\ q_r \end{pmatrix} = \frac{1}{4\epsilon} \begin{pmatrix} 1+\epsilon \\ -(1-\epsilon) \end{pmatrix}.$$

For any $\epsilon < 1$, **the green state becomes expensive and the red state becomes cheap.**

Δq^{SS}: Equilibrium state prices and supply shocks

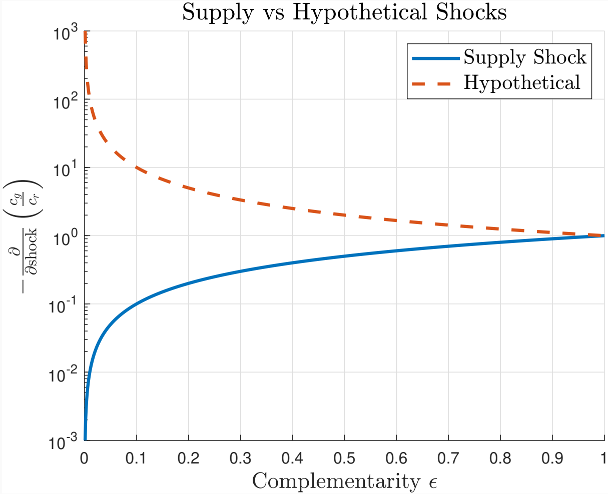
Impose market clearing. Then **equilibrium state prices** satisfy

$$q_g^*(s_g) = \frac{1}{2} \cdot \frac{1}{1 + \frac{s_g}{2}(1+\epsilon)} \qquad \text{and} \qquad q_r^*(s_g) = \frac{1}{2} \cdot \frac{1}{1 + \frac{s_g}{2}(1-\epsilon)}.$$

For any $\epsilon < 1$, **both state prices are decreasing in green supply s_g.** Hence **one is of the wrong sign.**

(Only exception is $\epsilon = 1$, in which case we recover Arrow securities.)

Optimal investor-level change in consumption ratio c_g/c_r (log scale)



(Preliminary) conclusions

Central properties of asset pricing model imply challenges for asset-level demand estimation:
(i) no arbitrage, (ii) preferences over cash flows, and (iii) variation based on supply shocks.

Potential remedies:

1. Certain assets may not be vulnerable to these issues. **Can check this using the payoff matrix**.
2. Constraints may help to solve the bundling issue (by analogy to "standard" IO).
3. Control variables can make the payoff matrix more diagonal (but also change the estimand).
4. Many supply shocks can help. But this requires strong stationarity (e.g., recovery theorems).
5. If you observe the demand curve, you don't need supply shocks. (e.g., Allen, Kastl and Wittwer).
6. ...