Initial Notes on Koijen and Yogo (2025): An Example on CAPM and No Arbitrage

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This note describes an example suggested by Narayana Kocherlakota. In the context of the CAPM, it illustrates that, while a valid stochastic discount factor (SDF) representation has a demand system representation, the converse need not hold: without additional restrictions, a given demand system may not have a valid SDF representation.

Suppose there are four equally likely states and three assets. Asset 1 has state-contingent gross returns $(1.1, 0.94, 1.1, 0.94)$ and asset 2 has state-contingent gross returns $(1.3, 0.98, 1.08, 1.30)$. The third asset is a safe one, with gross return 1 in all states. These three assets admit an arbitrage: buy 1 unit of asset 2, sell 0.5 units of asset 1 and sell 0.5 units of asset 3. This portfolio costs zero, and has a positive payoff in all states: $(0.25, 0.01, 0.03, 0.33)$.

Nonetheless, the CAPM holds in this setting. The market (maximal Sharpe ratio) portfolio of the two risky assets assigns a weight of 5.53% to asset 1 and a weight of 94.47% to asset 2. The expected return to the market portfolio is 15.7%.

Recall that we define β_1 (the beta of asset 1 with respect to the market portfolio) as:

$$\beta_1 = \frac{\mathrm{Cov}(r_1, r_{\mathrm{mkt}})}{\mathrm{Var}(r_{\mathrm{mkt}})} = 0.1274, \text{ whereas } \beta_2 = 1.0511.$$

We can verify that:

$$E(r_1) - r_3 - \beta_1(E(r_{\mathrm{mkt}}) - r_3) = 0.02 - 0.1274 \cdot 0.157 = 0,$$

$$E(r_2) - r_3 - \beta_2(E(r_{\mathrm{mkt}}) - r_3) = 0.165 - 1.0511 \cdot 0.157 = 0.$$

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The CAPM is basically the consequence of aggregating the first-order conditions of mean-variance optimization. But mean-variance optimization is well-defined even if the assets admit an arbitrage. Fundamentally, this is because mean-variance optimization does not impose monotonicity with respect to consumption/wealth.

KY19's reduced-form model would justify positive holdings of the above three assets on the basis of their differing characteristics. But no log-utility model with heterogeneous beliefs is consistent with the above returns. To see why, suppose that (m_1, m_2, m_3, m_4) represent positive state prices (or positive marginal utilities of consumption in period 2). Then:

$$1 = 1.1m_1 + 0.94m_2 + 1.1m_3 + 0.94m_4; \qquad \text{(Asset 1)}$$
$$1 = 1.3m_1 + 0.98m_2 + 1.08m_3 + 1.3m_4; \qquad \text{(Asset 2)}$$
$$1 = m_1 + m_2 + m_3 + m_4. \qquad \text{(Safe Asset)}$$

Multiply first and third equations 0.5 and subtract from the second equation. Then:

$$0 = 0.25m_1 + 0.01m_2 + 0.03m_3 + 0.33m_4 > 0.$$

So, there is no heterogeneous beliefs model that is consistent with these asset returns.