A Three-Assets Example with a Slight Perturbation

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April 21, 2025

We consider the payoff structure depicted in Table 1, where $y(1) = y(2) = 1$ and $\delta \in [0, 1)$ is a perturbation on the baseline model.

		State 1 (π_1)		State 2 (π_2)
		$\iota = g$ $(1-\rho)$	$\iota = r$ (ρ)	
Tree 1	green	$y(1) + \epsilon$	$y(1) - \epsilon$	0
	red	$y(1) - \epsilon$	$y(1) + \epsilon$	$\delta y(2)$
Tree 2		0		$(1-\delta)y(2)$

Table 1: The Description of the Payoff Structure.

When $\delta = 0$, the payoff structure is the same as in our baseline model and the resulting substitution matrix satisfies the symmetry assumptions in HHHKL. Hence, for $\delta = 0$, the diff-in-diff estimator proposed in HHHKL accurately recovers the *relative elasticity*. However, these assumptions are violated if $\delta > 0$. In this note, we show that small perturbations can lead to large biases when estimating the relative elasticity.

We follow HHHKL in defining the object of interest to be the observed change in relative portfolio shares over the difference of log price changes. We call this object $\hat{\mathcal{E}}_{g|r}$, and note that it is the proposed estimator in HHHKL. Following our analysis in the main paper, we decompose this *observed relative easticity* into the structural relative elasticity $\mathcal{E}_{g|r}$ and bias $\mathcal{B}_{g|r}$. Given the payoff structure above, this decomposition is

$$\underbrace{-\frac{\frac{d\omega_g}{de_g} - \frac{d\omega_r}{de_g}}{\frac{d\log p_g}{de_g} - \frac{d\log p_r}{de_g}}}_{\hat{\mathcal{E}}_{g|r}} = \underbrace{-\left(\frac{\partial \omega_g}{\partial p_g} p_g - \frac{\partial \omega_r}{\partial p_g} p_g\right)}_{\mathcal{E}_{g|r}} \underbrace{- \left(\frac{\partial \omega_g}{\partial p_g} p_g - \frac{\partial \omega_r}{\partial p_r} p_r\right) \frac{\frac{dp_r}{de_g}\frac{1}{p_r}}{\frac{dp_g}{de_g}\frac{1}{p_g} - \frac{dp_r}{de_g}\frac{1}{p_r}}}_{\mathcal{B}_{g|r}}.$$

To characterize the bias explicitly for our model economy, we compute demand functions for the perturbed economy analogous to Lemma 1 in the main paper. Extending the proof of Proposition 1 from that paper to these perturbed demand functions yields the following characterization. As in main paper, we place special emphasis on the limit $\epsilon \to 0$ where the inside assets become close substitutes, raising the importance of demand complementarities and spillovers.

Proposition 1 (Observed Relative Elasticity and Bias) *Let $\delta > 0$.*

1. *In the limit as $\epsilon \to 0$, the observed relative elasticity $\hat{\mathcal{E}}_{g|r}$ converges to:*

$$\pi_1 \cdot \frac{2 - \pi_1}{2(1 - \pi_1)} + \frac{\pi_1^2}{1 - \pi_1} \cdot \frac{1 - \delta}{\delta} \ll \infty, \tag{1}$$

   while the structural relative elasticity $\mathcal{E}_{g|r}$ goes to infinity.

2. *The bias $\mathcal{B}_{g|r}$ has the following properties: (i) it is non-negative; (ii) it is 0 if and only if $\delta = 0$ or $\epsilon = 1$; (iii) it is increasing in δ and decreasing in ϵ; and (iv) if $\delta > 0$, then it goes to infinity as $\epsilon \to 0$. In particular, when $\rho = \frac{1}{2}$, the bias is expressed as:*

$$\mathcal{B}_{g|r} = \frac{(1 - \pi_1)(1 - \epsilon^2)\delta}{2(2 - \delta)\epsilon^2} \cdot \frac{(2 - \delta)\pi_1 + \delta(1 - \pi_1)(1 + \epsilon^2)}{\delta(1 - \pi_1) + (\delta + 2\pi_1(1 - \delta))\epsilon^2}. \tag{2}$$

In the main paper, we showed that the bias in estimating the absolute elasticity is large when inside assets are close substitutes ($\epsilon \approx 0$), but small when they are poor substitutes ($\epsilon \to 1$). The proposition shows a similar result for the relative elasticity in the perturbed economy. Fixing δ at some small but strictly positive number, in the limit as $\epsilon \to 0$ the structural relative elasticity diverges but the estimated relative elasticity is bounded.

To get some rough intuition for magnitudes, Figure 1 illustrates the observed relative elasticity $\hat{\mathcal{E}}_{g|r}$ in the limit as $\epsilon \downarrow 0$ for various values π_1 as a function of δ. These parameters are important because they determine the degree to which the symmetry assumptions in HHHKL are violated. The violation is larger when the perturbation δ is large or when the probability of state 2 (where the asymmetry matters) is higher. The left panel shows the estimated relative elasticity as a function of δ **in the limit as $\epsilon \to 0$. The true structural relative elasticity in this limit is infinity.** The right panel of Figure 1 illustrates the bias $\mathcal{B}_{g|r} = \mathcal{E}_{g|r} - \hat{\mathcal{E}}_{g|r}$ as a function of ϵ for various values of δ. The other parameters are set at $\rho = \frac{1}{4}$ and $\pi_1 = \frac{1}{2}$.

For relative small values of δ, the estimated elasticity is thus orders of magnitude below the structural elasticity. However, the bias is relatively small when the assets are less substitute (large ϵ). Overall, we thus recover many of the same conclusions as in the main paper.

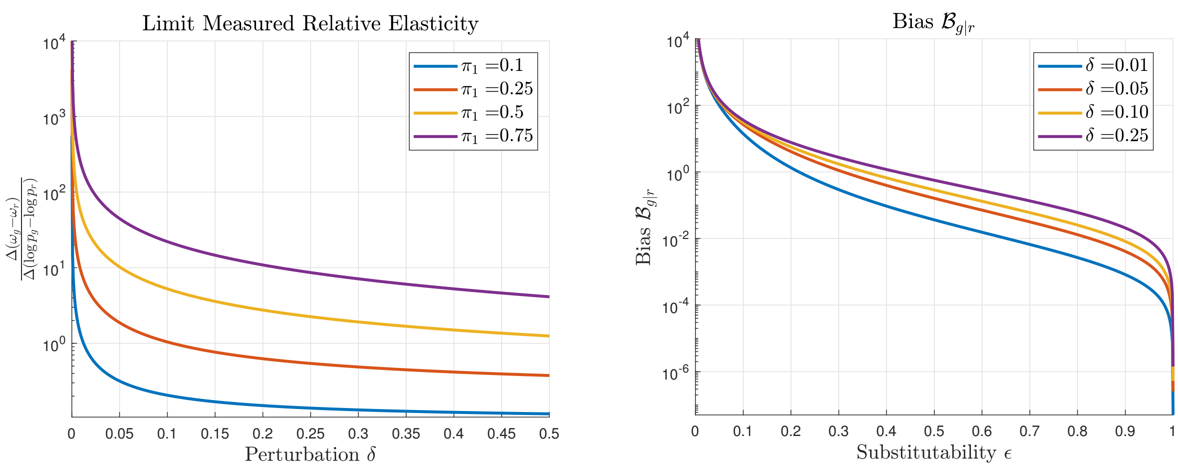

Figure 1: The Measured Elasticity in the Limit $\epsilon \downarrow 0$ (Left) and the Bias (Right).